Gravitational interaction of nuclei with mini black holes

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This research was supported by NSF grant #0647670, the Texas A&M Cyclotron Institute, and the Department of Energy grant DE-FG03-93ER40773.

Goals

•To find solutions to the Schrödinger Equation to describe the behavior of a particle around a mini black hole (BH)
• To calculate the absorption cross section and bound states of nuclei by the mini BH.

There are four basic types of black holes (BH):

•Static ($J = 0,\ Q = 0$)
•Static Charged ($J = 0,\ Q \neq 0$)
•Rotating ($J \neq 0,\ Q = 0$)
•Rotating Charged ($J \neq 0,\ Q \neq 0$)

These can be separated into:

• Star-collapsed BHs
• Mini BHs
• Galactic BHs

Mini Black Holes: An overview

Right after the Big Bang, density irregularities in primordial space could have allowed for mini BH (on the order of Planck length or larger) to form.

We consider quantum-interacting mini BH's about 19 orders smaller than the BH our sun could create.

$$r_s \approx 1\,\text{fm}, \quad M \sim 10^{16}\,\text{g}, \quad t_{ev} \sim 10^{10}\,\text{years},$$

The evaporation time depends on directly on mass and inversely on luminosity, making it possible to know the properties of mini BH's that could possibly be detected today.

The Coordinate System and Metric

In General relativity the observation depends on the adopted coordinate system.
A popular choice is the Schwarzschild metric

$$ds^2 = (1-\frac{r_s}{r})c^2dt^2 - \frac{dr^2}{1-\frac{r_s}{r}} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The Schwarzschild radius (event horizon) is:

$$r_s = \frac{2GM}{c^2}$$

The radial interval has an unphysical singularity at $r = r_s$

This singularity means that a particle will never reach the event horizon!

The Generalized Metric:

$$ds^2 = g_{00}(\mathbf{r},t)c^2dt^2 - 2g_{0i}cdtdx^i - g_{ji}(\mathbf{r},t)dx^j dx^i,$$

$$i, j = 1,2,3$$

To remove the Schwarzschild singularity we introduce Eddington-Finkelstein (E-F) coordinates with the contravariant and covariant tensors:

$$g^{\mu\nu} = \begin{pmatrix} 1+\frac{r_s}{r} & -\frac{r_s}{r} & 0 & 0 \\ -\frac{r_s}{r} & -(1-\frac{r_s}{r}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{r_s}{r} & -\frac{r_s}{r} & 0 & 0 \\ -\frac{r_s}{r} & -(1+\frac{r_s}{r}) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g_{\mu\nu}g^{\mu\nu} = I$$

In E-F metrics, we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (1-\frac{r_s}{r})c^2dt^2 - 2\frac{r_s c}{r}dt\,dr - (1+\frac{r_s}{r})dr^2$$

$$-r^2 d\Omega$$

This is a pseudo-Riemannian metric with determinant

$$g = \det g_{\mu\nu} = -r^4 \sin^2\theta < 0$$

The Klein-Gordon Equation

Klein-Gordon (KG) equation for a spinless particle in Minkowski (flat) spacetime is:

$$\hbar^2 g_{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \Psi = -m^2 c^2 \Psi$$

where $g_{\mu\nu}$ is the Minkowski flat-space metric tensor.

Its extension for a curved space is:

$$\hbar^2 \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \Psi = -m^2 c^2 \Psi$$

The Klein-Gordon Equation

Expanding, we get (in the spherical coordinate system):

$$\hbar^2 \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial x^0}\left(r^2 \sin\theta g^{00} \frac{\partial}{\partial x^0}\psi\right)+\frac{\partial}{\partial x^0}\left(r^2 \sin\theta g^{01} \frac{\partial}{\partial x^1}\psi\right)+\ldots\right]=-m^2c^2\psi$$

$$\psi \equiv \psi(t,r,\theta,\varphi)$$

Looking for stationary solution

$$\psi(t,\mathbf{r},\theta,\varphi) = \mathrm{e}^{i\frac{E}{\hbar}t}\,\psi(\mathbf{r},\theta,\varphi)$$

The Klein-Gordon Equation

$$\psi \equiv \psi(r,\theta,\varphi)$$

$$E^2 = m^2c^4 + p^2c^2$$

$$E = mc^2 + \frac{p^2}{2m}$$ Non-relativistic equation

$$\cancel{m^2c^2}\psi = (\cancel{m^2c^2} + p^2 + \frac{r_s}{r}m^2c^2 + \cancel{\frac{r_s}{r}p^2})\psi - i\hbar\frac{1}{c}\left[\left(\frac{r_s}{r}c\frac{\partial\psi}{\partial r} + \frac{r_s}{r}\frac{\partial\psi}{\partial r} - \frac{r_s}{r}\psi\right)\left(mc^2 + \cancel{\frac{p^2}{2m}}\right)\right] +$$

$$\hbar^2\left[\left(1 + \frac{r_s}{r}\right)\frac{\partial^2\psi}{\partial r^2} + \left(\frac{r_s}{r^2} - \frac{2}{r}\right)\frac{\partial\psi}{\partial r} - \left(\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi}\right]$$

We use nonrelativistic limit and neglect terms with v/c (and higher orders)

This gives us the Shrödinger equation in curved space.

Final radial Schrödinger Equation in curved space

To obtain a simpler equation, we use:

$$\psi_l(k,r) = \frac{U_l(k,r)}{kr}$$

We get the final Schrödinger Equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \tilde{V}_1(r) + \tilde{V}_2(r) + V^N(r) + \frac{l(l+1)}{2\mu r^2}\right]U_l(k,r) = E_p U_l(k,r)$$

$$V^N = -\frac{GMm}{r}$$

$$\tilde{V}_1(r) = -i(\hbar c)r_s\left[-\frac{1}{2r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]$$

Imaginary Potential

$$\tilde{V}_2(r) = \frac{1}{2}\frac{(\hbar c)^2}{mc^2}\frac{r_s}{r}\left[\frac{1}{r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right]$$

Generated by curved space (rs)

Absorption Cross-Section

Multiplying the Schrödinger equation by ψ^* and its complex conjugate by ψ and subtracting gives us particle conservation law

$$-i\hbar\int d\vec{s}\bullet\vec{j} = -\int d\vec{r}\nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = -i\hbar r_s^2\int d\Omega j_r = -i\hbar r_s^2 j_r(r_s)$$

$$\vec{j} = -i\frac{\hbar}{2m}\nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = -i\frac{\hbar}{2m}\left(\psi^* V\psi - \psi V^*\psi^*\right)$$

Where j is the current density-the number of particles crossing the unit area per second. To get the particle flux through into the BH, we use Gauss' theorem and integrate around the BH.

The RHS of the upper equation is proportional to the absorption cross-section. V_1 is the only part of the potential that doesn't cancel!

Absorption cross section

$$\sigma = i\frac{2m}{\hbar^2 k}\int d\vec{r}\left(\psi^* V_1 \psi - \psi V_1^* \psi^*\right) = i\frac{2m}{\hbar^2 k} 2\int d\vec{r}\psi^* V_1 \psi$$

$$= i\frac{4m}{\hbar^2 k}(4\pi)^2 \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\int_{r>r_s}^{\infty} dr \cdot r^2 \frac{U_l(k,r)}{kr}\left(-i\hbar\left[\frac{cr_s}{r}\frac{\partial}{\partial r}+\frac{r_s c}{2r^2}\right]\frac{U_l(k,r)}{kr}\right)$$

$$\sigma = 8\pi\frac{mc^2}{\hbar c}\frac{1}{k^3}\sum_{l=0}^{\infty}(2l+1)U_l^2(k,r)$$

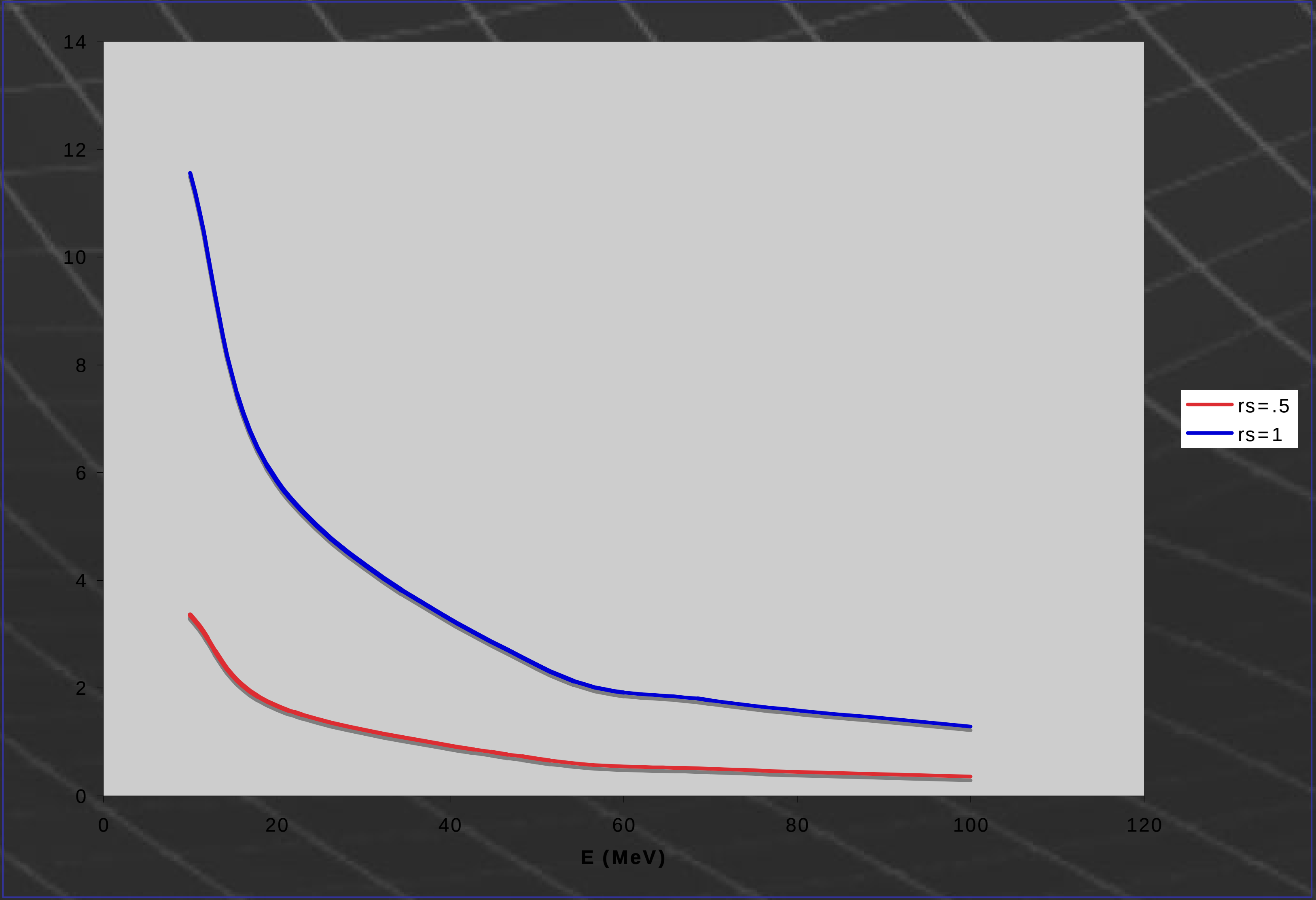
$$U_l(k,r) = e^{-\frac{\pi\eta}{2}}\frac{|\Gamma(l+1+i\eta)|}{2\Gamma(2l+2)}(2kr)^{l+1}e^{-ikr}{}_1F_1(l+1-i\eta, 2l+2, 2ikr)$$

Because V^N behaves like the Coulomb potential (V^C),
we can use and the Coulomb parameter (η) to parametrize our equation:

$$\eta^c = \frac{-z_1 z_2 e^2 m}{\hbar k} \rightarrow \eta = \frac{-GMm^2}{\hbar k} = \frac{-r_s}{\frac{\hbar}{mc}\sqrt{\frac{2E}{mc^2}}}$$

$$k = \frac{mv}{\hbar}$$

Absorption cross section



The mass gained by a BH going through the sun:

$$\rho = 1\frac{g}{cm^3}$$

$$m = 1.66 \cdot 10^{-24} g$$

$$N = \frac{1}{1.66 \cdot 10^{-23}} = 6.024 \cdot 10^{23} cm^{-3}$$

Number of protons in cm

Where ρ is the density of the sun, m is the mass of a proton, and 1/N is the volume/proton. The mean free path is

$$\lambda = \frac{1}{N\sigma} = \frac{1}{6.02 \cdot 10^{23} * 100 \cdot 10^{-24}} cm = 0.0166 cm$$

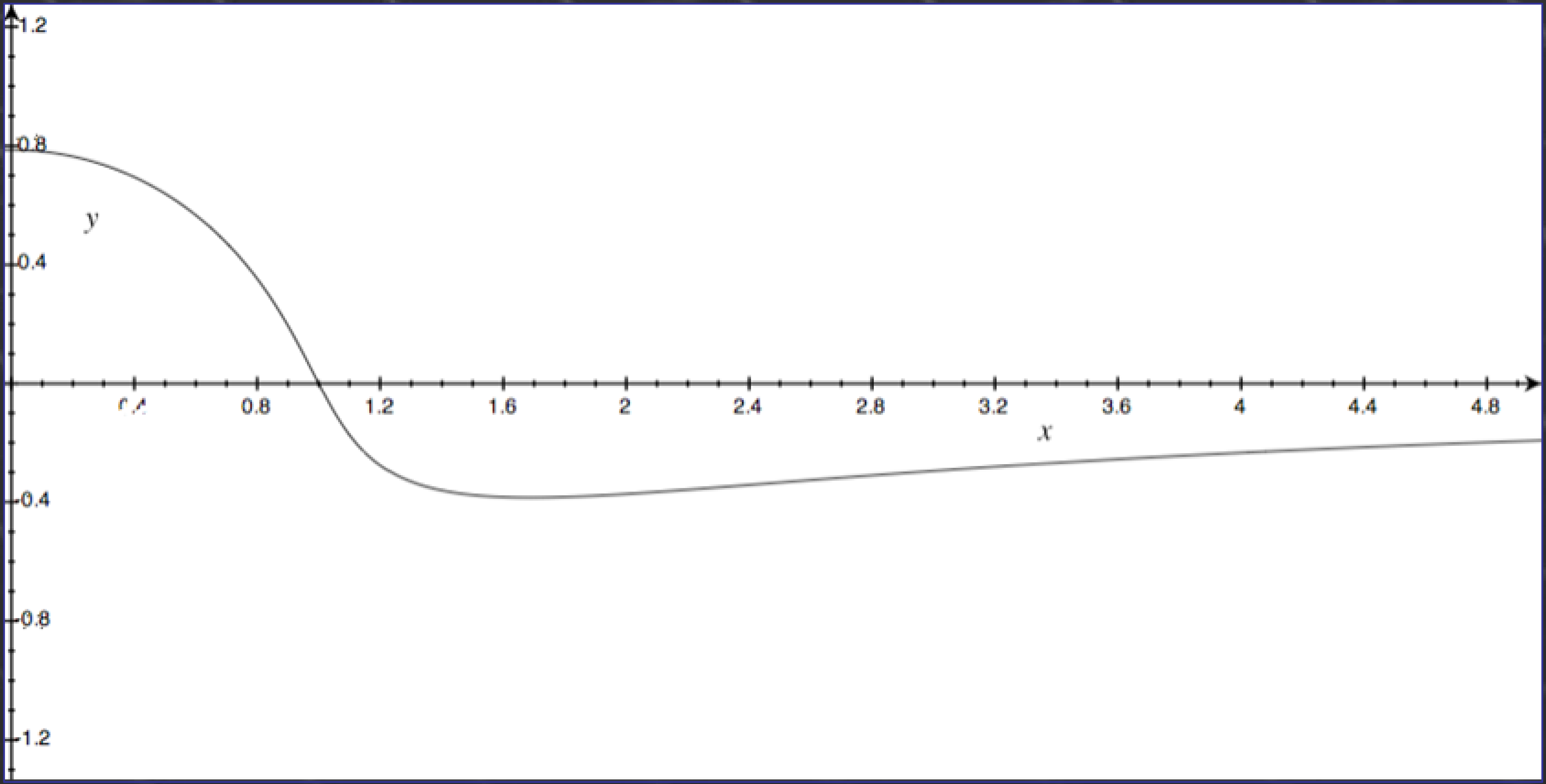
$$for\ \sigma = 100b, R = 7 \cdot 10^{10} cm$$

$$N = \frac{2R}{\lambda} = 8.44 \cdot 10^{12}$$

$$M = Nm = 1.4 \cdot 10^{-11} g$$

N is the number of particles in the path, M is the accumulated mass.
The gain is negligible!

The Schrödinger Equation



We see that the repulsive V_1+V_2 behaves more strongly than the attractive Newtonian potential. This means that the field of the BH is Repulsive at the origin.

Perturbation Approach

To simplify calculations, for r greater than r_s, we assume that V_1+V_2 can be considered as a perturbation to V^N

Doing this, we can get ψ as a solution to the Schrödinger equation.

At large distances, the potential behaves like

$$V_1 \approx i(931.54\,MeV)\frac{r_s}{r}\sqrt{\frac{2E}{mc^2}}$$

$$For\ E = 10\,MeV, \qquad V_1 \approx -i136\,MeV\frac{r_s}{r}$$

$$V^N \approx 470\,MeV\frac{r_s}{r}$$

$$V_2 \approx 19\,MeV\frac{r_s}{r}$$

So, perturbation approach may not be accurate enough.

Bound States

We continue our analogy to the Coulomb potential, and using the Hydrogen atom as our model we get the energy levels of a particle in a bound state with a mini BH:

$$E_n = \frac{mc^2 r_s^2}{8\left(\frac{\hbar}{mc}\right)^2 n^2}$$

With orbital radii:

$$r = a_0\sqrt{\frac{n^2(5n^2+1)}{2}}; \quad a_0 = 2\left[\frac{\hbar^2}{mc}\right]^2 \frac{1}{r_s} = 0.09\ fm$$

We consider r> r_s=1 fm and $|E_n|$ less than the particle's rest mass, and find that bound states of n=3 or higher are allowed.

These have energies < 290MeV.

To give this more physical context, we can compare terms using constants and some estimations.

To use a quantum mechanical approach, the Schwarzchild radius of the black hole and the wavelength of a particle around it must be of comparable size.

$$\lambda = \frac{\hbar}{mv} = \frac{\hbar}{mc}\sqrt{\frac{c^2}{v^2}} = 0.2118\, fm\sqrt{\frac{mc^2}{mv^2}} = 0.2118\, fm\sqrt{\frac{mc^2}{2E_{part}}}$$

$$*E_p = \frac{mv^2}{2} = \frac{p^2}{2m}$$

For a nucleon, the rest mass mc^2=939MeV, so for increasing energy:

$$\lambda = 0.2118\, fm\sqrt{\frac{939\, MeV}{2(0.01MeV)}} = 45.89\, fm$$

$$\lambda = 0.2118\, fm\sqrt{\frac{939\, MeV}{2(0.1MeV)}} = 14.51\, fm$$

$$\lambda = 0.2118\, fm\sqrt{\frac{939\, MeV}{2(1MeV)}} = 4.59\, fm$$

The smaller the BH (and the particle it interacts with), the larger its energy!

A future look at mini black holes

Cosmological relevancy:

CERN mini BH's from high-energy collisions hope to give us more information about the universe just after the Big Bang.

NASA's GLAST satellite has mini BH search on their horizon.

New Physics?

Mini BH's would violently evaporate the particles they've gradually accumulated in an amount of Hawking radiation inversely proportional to the mass of the BH. In this model, particles interact quantum mechanically in the relativistic space-time of a mini BH, making it a possible setting to study exciting new physics like quantum gravity.

Acknowledgments

Dr. A.M Mukhamedzhanov

Dr. Sherry Yenello

Larry May

Dr. Dave Toback

Paul Geffert

Cyclotron Institute, Texas A&M University

National Science Foundation & the Department of Energy

Schrödinger Equation

$$\hat{T} = \hat{T}_r + \hat{T}_{\theta,\varphi}; \quad \hat{T} = -\frac{h^2}{2m}\Delta$$

$$\hat{H}\psi = E_{part}\psi = (\hat{T} + V); \quad V = V^N + V_1 + V_2$$

$$\psi(r,\theta,\varphi) = \psi(r)\psi(\theta,\varphi); \quad \hat{H} = \hat{H}_r + \hat{H}_{\theta,\varphi}$$

$$\left(\hat{H}_r + \hat{H}_{\theta,\varphi}\right)\psi(r)\psi(\theta,\varphi) = \psi(\theta,\varphi)\hat{H}_r\psi(r) + \psi(r)\hat{H}_{\theta,\varphi}\psi(\theta,\varphi)$$

$$= \psi(\theta,\varphi)\hat{H}_r\psi(r) + \frac{l\left(l+1\right)}{2mr^2}\psi(r)\psi(\theta,\varphi)$$

$$= \psi(\theta,\varphi)\left[\hat{H}_r + \frac{l\left(l+1\right)}{2mr^2}\right]\psi(r)$$

Here, l is the orbital angular momentu
H is the Hamiltonian, E_p is the energy
Of the particle, and T is the kinetic ene
operator

The Schrödinger Equation in curved space

Since the generalized potential depends on l, we assign index l to the radial wave function: $\psi_l(r)$ and

$$\psi(\theta,\varphi)\left[\hat{T}_r + V_l(r)\right]\psi_l(r) = E_{part}\,\psi(\theta,\varphi)\psi_l(r)$$

$$V_l(r) = V(r) + \frac{l(l+1)}{2mr^2}$$

A positive centrifugal potential is <u>repulsive</u>, and represents a particle coming out of the BH.

Partial wave expansion of the wave function

$$\psi_{\vec{k}}(\mathbf{r}) = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta)\psi_l(k,r)$$

where $P_l(\cos\theta) = P_l\left(\frac{\vec{k}\bullet\vec{r}}{k\bullet r}\right)$ is the Legendre polynomial